

Algebraically special Einstein-Maxwell fields

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(Dated: 6 December 2016)

The Geroch-Held-Penrose formalism is used to re-analyse algebraically special non-null Einstein-Maxwell fields, aligned as well as non-aligned, in the presence of a possible non-vanishing cosmological constant. A new invariant characterization is given of the García-Plebański and Plebański-Hacyan metrics within the family of aligned solutions and of the Griffiths metrics within the family of the non-aligned solutions. As a corollary also the double alignment of the Debever-McLenaghan ‘class \mathcal{D} ’ metrics with non-vanishing cosmological constant is shown to be equivalent with the shear-free and geodesic behavior of their Debever-Penrose vectors.

The final publication is available at Springer via <http://dx.doi.org/10.1007/s10714-016-2171-x>

I. INTRODUCTION

In the quest for exact solutions of the Einstein-Maxwell equations,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \quad (1)$$

a large amount of research (see for example the reviews in^{21,40} has been devoted to the study of so called *aligned* Einstein-Maxwell fields, in which at least one of the principal null directions (PND’s) of the electromagnetic field tensor \mathbf{F} is parallel to a PND of the Weyl tensor, also called a Debever-Penrose direction, with main emphasis on the *doubly aligned* Petrov type D solutions, in which both real PND’s of \mathbf{F} are parallel to a corresponding double Weyl-PND. One of the prominent tools in these and related activities has been the Goldberg-Sachs theorem, which in its original form¹⁸ says that a vacuum space-time is algebraically special if and only if it contains a shear-free geodesic null congruence (in an adapted Newman-Penrose tetrad, ‘ $\Psi_0 = \Psi_1 = 0$ if and only if $\kappa = \sigma = 0$ ’). Goldberg and Sachs also proved that, if a space-time admits a complex null tetrad $(\mathbf{k}, \mathbf{\ell}, \mathbf{m}, \overline{\mathbf{m}})$ such that \mathbf{k} is shear-free and geodesic and $R_{ab}k^ak^b = R_{ab}k^am^b = R_{ab}m^am^b = 0$ (as is the case when \mathbf{k} is a PND of the electromagnetic field tensor), then the Weyl tensor is algebraically special, with \mathbf{k} being a multiple Weyl-PND. While, for a null Maxwell field^{29,38}, the Maxwell and Bianchi equations imply that both conditions $\Psi_0 = \Psi_1 = 0$ and $\kappa = \sigma = 0$ are trivially satisfied (also in the presence of a possible cosmological constant), the situation is less straightforward in the non-null case. It still is true^{18,39} that, when a PND \mathbf{k} of the electromagnetic field tensor is shear-free and geodesic, then the Weyl tensor is algebraically special, but the reverse no longer holds. A key property in this respect is the Kundt-Trümper theorem²⁶, which says that for an algebraically special aligned non-null Einstein-Maxwell space-time (with a possible non-0 cosmological constant and with \mathbf{k} the PND of \mathbf{F} coinciding with a multiple Weyl-PND) one necessarily has

$$\kappa(3\Psi_2 - 2|\Phi_1|^2) = 0 \text{ and } \sigma(3\Psi_2 + 2|\Phi_1|^2) = 0, \quad (2)$$

implying that the exceptional case $|\kappa|^2 + |\sigma|^2 \neq 0$ can only occur for Petrov types II or D with

$$\frac{3}{2}\Psi_2 = \pm|\Phi_1|^2. \quad (3)$$

Subsequent research has been concentrated on the doubly aligned Petrov type D cases, in which the Kundt-Trümper relations (2) also hold with κ and σ replaced by ν and λ . This culminated in the complete integration of the field equations for the Petrov type D doubly aligned non-null Einstein-Maxwell fields, with a possible non-0 cosmological constant:

- When $\frac{9}{4}\Psi_2^2 - |\Phi_1|^4 \neq 0$, both real PND’s are geodesic and shear-free ($\kappa = \nu = \sigma = \lambda = 0$) and, following Debever and McLenaghan¹⁰, I will refer to the corresponding set of solutions as the ‘class \mathcal{D} ’ space-times. They all admit at least a two-dimensional isometry group and count among their most famous members the Reissner-Nordström

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and Kerr-Newman metrics. Several authors^{5–12,14,15,22,24,33,34} have independently contributed to the determination of class \mathcal{D} , beginning with Carter’s seminal study of the separability of the Hamilton-Jacobi and Klein-Gordon equations and culminating in the discovery³³ of Plebański and Demiański’s 7-parameter metric for the non-null orbit solutions and García’s construction^{14,15} of a single metric form for both the non-null and null orbit solutions.

- The case $\frac{3}{2}\Psi_2 = |\Phi_1|^2$ was fully integrated in³⁵. In the resulting ‘Plebański-Hacyan space-times’ \mathbf{k} and ℓ are respectively non-geodesic ($\kappa \neq 0$) and geodesic ($\nu = 0$), while both are shear-free ($\sigma = \lambda = 0$) and have vanishing complex divergence ($\rho = \mu = 0$). The metric is given by

$$ds^2 = 2d\zeta d\bar{\zeta} + 2dudv + [\Lambda u^2 + \zeta \bar{F}(v) + \bar{\zeta} F(v)]dv^2, \quad (4)$$

which, for an arbitrary function $F(v)$ ⁴³ does not admit any isometries and only has an electromagnetic energy-momentum tensor of the correct sign when $\Lambda < 0$.

- The case $\frac{3}{2}\Psi_2 = -|\Phi_1|^2$, which was overlooked in³⁵, was dealt with in¹⁶. Its unique solutions are the ‘García-Plebański metrics’,

$$ds^2 = -\frac{1}{\Lambda} \left(e^{2z} \omega^{12} + e^{-2z} \omega^{22} + dz^2 - 4(\cosh z)^2 \omega^{32} \right), \quad (5)$$

with

$$\omega^1 + i\omega^2 = 2 \frac{e^{iu}}{1 - \zeta \bar{\zeta}} d\zeta \text{ and } \omega^3 = du - i \frac{\zeta d\bar{\zeta} - \bar{\zeta} d\zeta}{1 - \zeta \bar{\zeta}}. \quad (6)$$

The corresponding space-times admit a 3 dimensional group of isometries; both \mathbf{k} and ℓ are geodesic ($\kappa = \nu = 0$), shearing ($\sigma \lambda \neq 0$) and twisting, but non-expanding ($\rho, \mu \in i\mathbb{R}$). Also this metric can only describe an Einstein-Maxwell space-time with an electromagnetic energy-momentum tensor of the correct sign when $\Lambda < 0$.

In this paper I will have a closer look at the full set of algebraically special cases, both aligned (and hence by (3) necessarily of Petrov type II) and non-aligned ones. First we will see that the García-Plebański metrics are the unique members of the class of algebraically special and aligned Einstein-Maxwell solutions for which the null direction \mathbf{k} is shearing. While doing so, I will correct an error in²⁵ (cited also in⁴⁰ p. 409), claiming that, at least for $\Lambda = 0$, the case $\kappa = 0 \neq \sigma$ admits no solutions. This is true indeed but, as will become clear in section 2, the proof requires a much more subtle argumentation than the one presented in^{25,44}. The error occurs after relation (2.6) when the case $\bar{\rho} = -\rho$ is dismissed by remarking that it “leads to $\rho = 0$ ”. Most likely this conclusion was prematurely arrived at by inspection of the Newman-Penrose equation corresponding to our GHP equation (87): with $\kappa = \epsilon + \bar{\epsilon} = \Phi_0 = 0$ this equation reads $D\rho = \rho^2 + \sigma\bar{\sigma}$, the real and imaginary parts of which only allow one to conclude that $\rho = \pm i|\sigma|$ and $D\rho = D\sigma = 0$. In section 2 a correct proof of Kozarzewski’s no-go claim will be provided, generalizing it to the case $\Lambda \geq 0$ and showing that for $\Lambda < 0$ the only allowed solutions are the doubly aligned García-Plebański metrics with $\rho = \pm i|\sigma| \neq 0$.

In section 3 I will consider the algebraically special and aligned Einstein-Maxwell fields for which the null direction \mathbf{k} is non-geodesic. The general solution in this family so far is not known, but, remarkably, the Plebański-Hacyan metrics exhaust the sub-family characterised by the vanishing of the complex divergence of \mathbf{k} .

Finally I prove in section 4 that an algebraically special Einstein-Maxwell solution possessing a shear-free and geodesic multiple Weyl-PND which is *not* a PND of \mathbf{F} necessarily has vanishing cosmological constant and I give a characterization of the sub-class of the Griffiths²⁰ solutions. As a corollary of this theorem it follows that the ‘class \mathcal{D} ’ metrics¹⁰ are the unique Petrov type D Einstein-Maxwell solutions for which the real Weyl-PND’s are both geodesic and shear-free and for which the cosmological constant is non-vanishing: in other words, the double alignment condition of the class \mathcal{D} metrics with non-vanishing cosmological constant is a consequence of their multiple Weyl-PND’s being geodesic and shear-free. Whether this property persists when $\Lambda = 0$ is at present still an open problem, with only the Kundt case ($\rho = 0$) so far having been dealt with.

In order to study these and related issues, linking kinematic properties of certain invariantly defined null directions (such as being geodesic and/or non-shearing) to algebraic properties of the electromagnetic field tensor or of the Weyl tensor, it is natural to use the Geroch-Held-Penrose (GHP) formalism¹⁷. In this formalism only a pair of null *directions* is singled out at each point rather than an entire null tetrad, as is the case in the Newman-Penrose³¹ formalism. The resulting formalism is covariant with respect to rotations of the spatial basis vectors and boosts of the

real null directions and, as such, is ‘halfway’ between a fully covariant approach and the NP spin-coefficient approach and leads to considerably simpler equations with fewer complex variables.

Throughout I will use the sign conventions and notations of Kramer §7.4, with the tetrad basis vectors taken as $\mathbf{k}, \mathbf{\ell}, \mathbf{m}, \mathbf{\bar{m}}$ with $-k^a \ell_a = 1 = m^a \bar{m}_a$. However, in order to ease comparison with the (more familiar) Newman-Penrose formalism, I will write primed variables, such as κ', σ', ρ' and τ' , as their NP equivalents $-\nu, -\lambda, -\mu$ and $-\pi$. For completeness the resulting weights, commutators, GHP, Bianchi and Maxwell equations are presented in an Appendix.

Finally note that, at least when the electromagnetic field is non-null, the pair of null directions $\mathbf{k}, \mathbf{\ell}$ can always be invariantly defined, aligning for example \mathbf{k} with a PND of \mathbf{F} and null-rotating about \mathbf{k} such that Φ_1 is the only non-vanishing component of the Maxwell spinor. Obviously this choice is not unique (for example in the non-aligned case it will be preferable to align \mathbf{k} with the multiple Weyl-PND and to null-rotate about \mathbf{k} such that $\Phi_1 = 0$ and $\Phi_0 \Phi_2 \neq 0$), but it is important to realise that any ensuing well-weighted GHP relations, such as $\kappa = 0, \rho = 0, \dots$ are automatically *geometrically invariant statements*.

II. ALIGNED ELECTROVACS WITH A SHEARING MULTIPLE DEBEVER-PENROSE VECTOR

Let us first, for the sake of completeness, re-confirm the well-known fact^{18,39} that if a PND \mathbf{k} of the electromagnetic field tensor \mathbf{F} is shear-free and geodesic, then the Weyl tensor is algebraically special with \mathbf{k} being the multiple Weyl-PND: with \mathbf{k} a PND of \mathbf{F} ($\Phi_0 = 0$) satisfying $\kappa = \sigma = 0$, it follows from (88) that $\Psi_0 = 0$ and hence \mathbf{k} is also a Weyl-PND. When \mathbf{F} is non-null we can null-rotate about \mathbf{k} such that also $\Phi_2 = 0$. The integrability conditions for the Maxwell equations (94, 95) simplify then with (89, 91) to $\Phi_1 \Psi_1 = 0$. It follows that $\Psi_1 = 0$ and hence \mathbf{k} is a multiple Weyl-PND.

Now let us consider the reverse situation and assume that the Weyl tensor is algebraically special, with the multiple Weyl-PND \mathbf{k} being also a PND of the non-null electromagnetic field tensor: by means of a suitable null rotation about \mathbf{k} the null tetrad can be chosen such that $\Psi_0 = \Psi_1 = \Phi_0 = \Phi_2 = 0$. The Maxwell equations (94, 95') and (95) become

$$\mathbb{P}\Phi_1 = 2\rho\Phi_1, \quad \bar{\partial}\Phi_1 = 2\tau\Phi_1, \quad \bar{\partial}'\Phi_1 = -2\pi\Phi_1, \quad (7)$$

while the Bianchi equations (96, 97) yield the Kundt-Trümper relations (2). In the present paragraph we consider the case where \mathbf{k} is shearing and hence

$$\kappa = 0, \quad \Psi_2 = -\frac{2}{3}|\Phi_1|^2. \quad (8)$$

Bianchi equation (98) simplifies now to $|\Phi_1|^2(\rho + \bar{\rho}) = 0$, implying that \mathbf{k} is non-expanding. From the real part of GHP equation (87) it follows that ρ is related to σ by

$$\rho = \pm i|\sigma| \quad (9)$$

and hence, by (87, 88, 89)

$$\mathbb{P}\rho = 0, \quad \mathbb{P}\sigma = 0, \quad \mathbb{P}\tau = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi). \quad (10)$$

From (99,99') one finds then

$$\Psi_3 = -\frac{2}{3} \frac{|\Phi_1|^2}{\sigma} (2\tau + \bar{\pi}), \quad (11)$$

and

$$\mathbb{P}\pi = 2(\bar{\pi}\bar{\sigma} - \bar{\tau}\rho). \quad (12)$$

Acting with the $[\bar{\partial}, \mathbb{P}]$ commutator on Φ_1 now leads to an expression for $\bar{\partial}\rho$, which, together with (91) and the $\bar{\partial}$ derivative of (9), yields

$$\bar{\partial}\rho = \sigma(\bar{\tau} + 2\pi) + 2\rho\tau, \quad (13)$$

$$\bar{\partial}'\sigma = \sigma(\bar{\tau} + 2\pi), \quad (14)$$

$$\bar{\partial}\sigma = \sigma(3\tau - 2\bar{\pi}) - 2\rho \frac{\sigma}{\bar{\sigma}} (\bar{\tau} + 2\pi). \quad (15)$$

The remaining integrability conditions for the Maxwell equations can then be written as

$$\mathbb{P}'\tau + \bar{\partial}\mu = \rho\bar{\nu} + \bar{\lambda}\pi, \quad (16)$$

$$\mathbb{P}'\pi - \bar{\partial}'\mu = \lambda\tau - \nu\rho - \mu\bar{\tau} - \bar{\mu}\pi, \quad (17)$$

$$\bar{\partial}\pi + \bar{\partial}'\tau = \rho(\mu + \bar{\mu}), \quad (18)$$

$$\mathbb{P}\mu + \mathbb{P}'\rho = \pi\bar{\pi} - \tau\bar{\tau}. \quad (19)$$

Next we apply the $[\bar{\partial}, \mathbb{P}]$ commutator to σ , to obtain an expression for τ ,

$$\tau = \frac{1}{3}\bar{\pi} + \frac{4}{3}\frac{\rho\pi}{\sigma}, \quad (20)$$

which together with its \mathbb{P} derivative leads to

$$\sigma\pi - \rho\bar{\pi} = 0. \quad (21)$$

This suggests the definition of an auxiliary variable S (by (9) one has $|S| = 1$, while $\text{weight}(S) = [2, -2]$) such that $\sigma = S\rho$ and enabling us to conclude from (21) that

$$\tau = -\bar{\pi}, \quad \bar{\pi} = S\pi. \quad (22)$$

By (10,13–15) one has then

$$\mathbb{P}S = 0, \quad \bar{\partial}S = -S(3\bar{\pi} - S\pi), \quad \bar{\partial}'S = 3S\pi - \bar{\pi}, \quad (23)$$

after which (97', 98, 98') yield

$$\bar{\partial}\pi = S\pi^2 + 2\rho(\mu + \bar{\mu}) - \frac{3}{2}S\rho^2\Psi_4|\Phi_1|^{-2}, \quad (24)$$

$$\bar{\partial}'\pi = -\pi^2 - 2S^{-1}\rho(\mu + \bar{\mu}) - \frac{3}{2}S^{-2}\rho^2\bar{\Psi}_4|\Phi_1|^{-2}, \quad (25)$$

$$\mathbb{P}\Psi_4 = \rho(\Psi_4 - S^{-2}\bar{\Psi}_4) - \frac{4}{3}S^{-1}(\mu + \bar{\mu})|\Phi_1|^2. \quad (26)$$

Herewith (18) reduces to

$$2(\mu + \bar{\mu})|\Phi_1|^2 + \rho(S^{-1}\bar{\Psi}_4 - S\Psi_4) = 0, \quad (27)$$

while expressing that $\mathbb{P}'(S\bar{S}) = 0$ (with $\mathbb{P}'S$ evaluated from (92)) and simplifying the result by means of (27), one finds that

$$\mu + \bar{\mu} = 0 \quad (28)$$

(i.e. also ℓ is non-expanding) and hence $\bar{\Psi}_4 = S^2\Psi_4$.

Furthermore, calculating $\mathbb{P}'\rho$ from (93) and expressing that $\mathbb{P}'(\rho + \bar{\rho}) = 0$ gives us an equation from which we can obtain Ψ_4 , $\mathbb{P}'\rho$ and $\mathbb{P}'S$:

$$\Psi_4 = \frac{|\Phi_1|^2}{18\rho^2} [24\pi^2 + 6\rho(\lambda - S^{-2}\bar{\lambda}) - S^{-1}(8|\Phi_1|^2 + 12\rho\mu - R)], \quad (29)$$

$$\mathbb{P}'\rho = -\frac{\rho}{2}(S\lambda + S^{-1}\bar{\lambda}), \quad (30)$$

$$\mathbb{P}'S = S \left[-2\mu + S\lambda - S^{-1}\bar{\lambda} + \frac{1}{\rho}(2S\pi^2 - \frac{2}{3}|\Phi_1|^2 + \frac{1}{12}R) \right]. \quad (31)$$

Next we solve (96', 97') together with (19) and the $[\mathbb{P}', \mathbb{P}]\rho$, $[\bar{\partial}, \mathbb{P}']S$ commutator relations for $\mathbb{P}'\pi$, $\bar{\partial}\mu$, $\mathbb{P}\mu$, $\mathbb{P}\lambda$, $\bar{\partial}\lambda$ and $\bar{\partial}'\lambda$:

$$\mathbb{P}'\pi = -\frac{3}{2}S\lambda\pi + \frac{1}{2}S^{-1}\rho\bar{\nu} + S^{-1}\bar{\lambda}\pi + \frac{1}{2}\mu\pi - \frac{5}{2}\nu\rho + (\frac{4}{3}|\Phi_1|^2 - 3S\pi^2 - \frac{1}{8}R)\frac{\pi}{\rho}, \quad (32)$$

$$\bar{\partial}\mu = -\frac{5}{2}S\rho\nu - \frac{1}{2}\lambda\pi S^2 - \frac{3}{2}S\mu\pi + \frac{3}{2}\rho\bar{\nu} + \bar{\lambda}\pi + (\frac{2}{3}|\Phi_1|^2 - S\pi^2 - \frac{1}{24}R)\frac{S\pi}{\rho}, \quad (33)$$

$$\mathbb{P}\mu = \frac{1}{2}\rho(S\lambda + S^{-1}\bar{\lambda}), \quad (34)$$

$$\mathbb{P}\lambda = \frac{1}{2}\rho(\lambda + S^{-1}\bar{\lambda}) - 2\pi^2 - \frac{1}{12}S^{-1}(R - 8|\Phi_1|^2), \quad (35)$$

$$\bar{\partial}\lambda = -\frac{5}{2}(\mu\pi + \nu\rho) + \frac{3}{2}S\lambda\pi + \frac{7}{2}S^{-1}\rho\bar{\nu} + (S\pi^2 - \frac{8}{3}|\Phi_1|^2 + \frac{1}{24}R)\frac{\pi}{\rho}, \quad (36)$$

$$\bar{\partial}'\lambda = \frac{1}{2}S^{-1}(\mu\pi + \rho\nu) + \frac{3}{2}S^{-2}(3\rho\bar{\nu} - \bar{\lambda}\pi) - 3\lambda\pi + S^{-1}(3S\pi^2 - \frac{8}{3}|\Phi_1|^2 + \frac{1}{8}R)\frac{\pi}{\rho}, \quad (37)$$

after which an expression for the spin coefficient μ follows from the $[\bar{\partial}', \bar{\partial}]S$ commutator relation,

$$\mu = \frac{1}{2}(S\lambda - S^{-1}\bar{\lambda}) + \frac{1}{36\rho}(24S\pi^2 - 32|\Phi_1|^2 + R). \quad (38)$$

Of the Maxwell integrability conditions there only remains now the $[\bar{\partial}', \mathbb{P}']\Phi_1$ relation, namely

$$6(S\lambda + S^{-1}\bar{\lambda})\rho\pi + \pi(8|\Phi_1|^2 - 24S\pi^2 - R) - 36\rho^2(\nu - S^{-1}\bar{\nu}) = 0, \quad (39)$$

which with GHP equation (91') gets simplified to the key algebraic equation

$$\pi(8|\Phi_1|^2 - 24S\pi^2 - R) = 0. \quad (40)$$

Herewith all Bianchi equations and GHP equations (except those involving the derivatives of ν) are identically satisfied. At this stage the Weyl spinor components are given by, using (38) to simplify (29),

$$\Psi_0 = \Psi_1 = 0, \quad (41)$$

$$\Psi_2 = -\frac{2}{3}|\Phi_1|^2, \quad (42)$$

$$\Psi_3 = \frac{2}{3}\frac{\pi}{\rho}|\Phi_1|^2, \quad (43)$$

$$\Psi_4 = \frac{|\Phi_1|^2}{27\rho^2}S^{-1}(24S\pi^2 + 4|\Phi_1|^2 + R). \quad (44)$$

I now show that the case $\pi \neq 0$ is inconsistent, while the case $\pi = 0$ leads to the García-Plebański metrics.

II.1. $\pi \neq 0$

When $\pi \neq 0$ we can use (40) to rewrite (39) as

$$\pi(S\lambda + S^{-1}\bar{\lambda}) - 6\rho(\nu - S^{-1}\bar{\nu}) = 0, \quad (45)$$

while (38) becomes

$$\mu = \frac{1}{2}(S\lambda - S^{-1}\bar{\lambda}) - \frac{2}{3}|\Phi_1|^2\rho^{-1}. \quad (46)$$

Taking the $\bar{\partial}$ derivative of (46) and eliminating π from the resulting equation and (40), we also find

$$3\rho(S\lambda - \bar{S}\bar{\lambda}) - 4|\Phi_1|^2 = 0 \quad (47)$$

and hence, by (46), $\mu = 0$. The \mathbb{P} derivative of (47) yields then $S\lambda + S^{-1}\bar{\lambda} = 0$ and hence, by (47), $\lambda = \frac{2}{3}|\Phi_1|^2S^{-1}\rho^{-1}$. Substituting this in the expression (36) for $\bar{\partial}\lambda$ one obtains $\nu = \frac{2}{3}|\Phi_1|^2\pi\rho^{-2}$, which together with GHP equation (88') leads to the contradiction $|\Phi_1|^2\pi = 0$.

II.2. $\pi = 0$

Substituting $\pi = 0$ in (24,44) one obtains

$$R (= 4\Lambda) = -4|\Phi_1|^2 \quad (48)$$

and

$$\bar{\nu} - 5S\nu = 0, \quad (49)$$

whence $\nu = 0$. While (48) proves Kozarzewski's no-go claim for $\Lambda = 0$, it also generalizes it to the case $\Lambda \geq 0$. Furthermore, when $\Lambda < 0$ the only non-0 Weyl spinor component is $\Psi_2 = R/6$ and we are in the doubly aligned situation with both Weyl-PND's \mathbf{k} and $\mathbf{\ell}$ being geodesic and non-expanding. The field equations and Maxwell equations have been completely integrated in this case by García and Plebański¹⁶ and the resulting metric is given by (5), admitting a 3D isometry group.

III. ALIGNED ELECTROVACS WITH A NON-GEODESIC MULTIPLE DEBEVER-PENROSE VECTOR

We again consider the case where the Weyl tensor is algebraically special, with the multiple Weyl-PND \mathbf{k} being also a PND of the non-null electromagnetic field tensor, but now we take $\kappa \neq 0$ and hence, by (2),

$$\sigma = 0 \text{ and } \Psi_2 = \frac{2}{3}|\Phi_1|^2. \quad (50)$$

Bianchi equation (99) now reduces to $|\Phi_1|^2(\tau - \bar{\pi}) = 0$, whence $\tau = \bar{\pi}$, after which (98) implies

$$\Psi_3 = \frac{2}{3} \frac{2\rho - \bar{\rho}}{\kappa} |\Phi_1|^2, \quad (51)$$

showing that solutions cannot be of Petrov type III or N when $\rho \neq 0$.

GHP equations (88,91,92) and Bianchi equation (98) yield then

$$\bar{\partial}\kappa = 0, \quad (52)$$

$$\bar{\partial}\rho = \bar{\pi}(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}), \quad (53)$$

$$\bar{\partial}'\rho = -2\pi\bar{\rho} - 2\bar{\kappa}\bar{\mu}, \quad (54)$$

$$\bar{\partial}'\pi = \pi^2 + \lambda\bar{\rho} - \nu\bar{\kappa}. \quad (55)$$

The integrability conditions for the Maxwell equations furthermore give an expression for $\mathbb{P}\pi$,

$$\mathbb{P}\pi = \bar{\kappa}(2\bar{\mu} - \mu) + 2\pi\bar{\rho}, \quad (56)$$

together with four extra relations

$$\bar{\partial}\mu + \mathbb{P}'\bar{\pi} = \bar{\nu}\rho + \bar{\lambda}\pi, \quad (57)$$

$$\bar{\partial}'\mu - \mathbb{P}'\pi = \pi(\bar{\mu} + \mu) - \lambda\bar{\pi} + \nu\rho, \quad (58)$$

$$\bar{\partial}\pi + \bar{\partial}'\bar{\pi} = \rho\bar{\mu} - \bar{\rho}\mu, \quad (59)$$

$$\mathbb{P}\mu + \mathbb{P}'\rho = 0, \quad (60)$$

with which GHP equations (89,92') imply

$$\mathbb{P}'\kappa = \kappa(2\mu - \bar{\mu}), \quad (61)$$

$$\mathbb{P}\lambda = \lambda(\rho + \bar{\rho}) + 2\pi^2 - 2\nu\bar{\kappa}. \quad (62)$$

So far it has not been possible to complete the analysis of this case. However, it is easy to see that the Plebański-Hacyan metrics completely exhaust the non-diverging subfamily of solutions ($\rho = 0$) and have $\Lambda < 0$, whereas for $\Lambda \geq 0$ no solutions exist. In fact, when $\rho = 0$ (54) and $\kappa \neq 0$ imply that also the second PND ℓ of the electromagnetic field is non-diverging ($\mu = 0$): herewith (51) and (99') show that $\Psi_3 = \Psi_4 = 0$, so that we are in the doubly aligned Petrov type D case. This suffices already to conclude that the only possible solutions are the Plebański-Hacyan metrics, as this case was completely integrated in³⁵, modulo the error corrected in¹⁶. However, it is instructive to provide a short and coordinate independent proof of the essential step in³⁵, namely the part in which the authors prove that $\Gamma_{423} = 0$, leading to the conclusion that ℓ not only is shear-free ($\lambda = 0$ being an immediate consequence of (97')), but also is geodesic ($\nu = 0$):

First observe that the $[\bar{\partial}, \mathbb{P}']$ commutator applied to κ implies $\bar{\nu}\mathbb{P}\kappa = 0$. Assuming $\nu \neq 0$ leads then to an inconsistency, as we would have

$$\mathbb{P}\kappa = 0, \quad \bar{\partial}'\kappa = \bar{\kappa}\bar{\pi} - \kappa\pi, \quad (63)$$

with the second relation being obtained from (87). Herewith the $[\mathbb{P}, \mathbb{P}']\kappa$ commutator relation yields

$$\kappa(\bar{\kappa}\bar{\nu} + 3\kappa\nu - 6\pi\bar{\pi} + \frac{1}{6}R - \frac{20}{3}|\Phi_1|^2) + 2\bar{\kappa}\bar{\pi}^2 = 0, \quad (64)$$

which, with $\nu\bar{\kappa} = \pi^2$ (obtained from (92')) simplifies to

$$|\kappa|^2(40|\Phi_1|^2 - R) - 18(\kappa\pi - \bar{\kappa}\bar{\pi})^2 = 0. \quad (65)$$

On the other hand (93'-93) reads

$$|\kappa|^2(8|\Phi_1|^2 + R) - 6(\kappa\pi - \overline{\kappa\pi})^2 = 0, \quad (66)$$

which, when added to (65), results in

$$2|\kappa\Phi_1|^2 - (\kappa\pi - \overline{\kappa\pi})^2 = 0. \quad (67)$$

The left hand side being positive definite, this shows that the case $\nu \neq 0$ is inconsistent. We therefore have $\nu = \pi = 0$ and hence, by GHP equation (93), $R = 4\Lambda = -8|\Phi_1|^2 < 0$.

IV. ELECTROVACS WITH A SHEAR-FREE AND GEODESIC MULTIPLE DEBEVER-PENROSE VECTOR

We now consider algebraically special (non-conformally flat) Einstein-Maxwell fields with a possible non-zero cosmological constant for which the multiple Weyl-PND \mathbf{k} is geodesic and shear-free ($\Psi_0 = \Psi_1 = \kappa = \sigma = 0$) and for which \mathbf{k} is not parallel to a PND of \mathbf{F} ($\Phi_0 \neq 0$). Choosing a null-rotation about \mathbf{k} such that $\Phi_1 = 0$, it follows that $\Phi_2 \neq 0$ ⁴⁵.

The Maxwell equations (94,95) and Bianchi equations (96-99) yield then

$$\bar{\partial}\Phi_0 = 0, \quad (68)$$

$$\bar{\partial}'\Phi_0 = -\pi\Phi_0, \quad (69)$$

$$\mathbb{P}\Phi_0 = 0, \quad (70)$$

$$\mathbb{P}'\Phi_0 = -\mu\Phi_0, \quad (71)$$

$$\bar{\partial}\Phi_2 = -\nu\Phi_0 + \tau\Phi_2, \quad (72)$$

$$\mathbb{P}\Phi_2 = -\lambda\Phi_0 + \rho\Phi_2, \quad (73)$$

$$\bar{\partial}\Psi_2 = -\pi\Phi_0\overline{\Phi_2} + 3\tau\Psi_2, \quad (74)$$

$$\mathbb{P}\Psi_2 = \mu|\Phi_0|^2 + 3\rho\Psi_2, \quad (75)$$

after which the commutators $[\bar{\partial}', \bar{\partial}]$, $[\bar{\partial}', \mathbb{P}]$, $[\bar{\partial}, \mathbb{P}']$ and $[\mathbb{P}', \mathbb{P}]$ applied to Φ_0 give

$$\bar{\partial}\pi = (3\rho - \bar{\rho})\mu - 2\Psi_2 + \frac{R}{12}, \quad (76)$$

$$\mathbb{P}\pi = 3\rho\pi, \quad (77)$$

$$\bar{\partial}\mu = \bar{\lambda}\pi + 3\mu\tau, \quad (78)$$

$$\mathbb{P}\mu = \pi(\bar{\pi} + 3\tau) + 2\Psi_2 - \frac{R}{12}. \quad (79)$$

Herewith GHP equation (93') becomes a simple algebraic equation for Ψ_2 ,

$$\Psi_2 = \rho\mu - \tau\pi + \frac{R}{12}, \quad (80)$$

the \mathbb{P} derivative of which (using (78,79,87,89)) results in $\rho R = 0$.

As $\rho = 0$ would imply $\Phi_0 = 0$, this shows that an algebraically special Einstein-Maxwell solution possessing a shear-free and geodesic multiple Weyl-PND which is *not* a PND of \mathbf{F} necessarily has a vanishing cosmological constant⁴⁶. The corresponding class of solutions is non-empty: imposing the additional restriction that $\pi = 0$ one can deduce that $\bar{\rho} = \rho$ and $\bar{\mu} = \mu$, together with a $[-2, 0]$ -weighted relation $\Psi_3 = \rho\nu + \mu\bar{\tau} - \tau\lambda$. It is then straightforward to construct a Newman-Penrose null-tetrad with the additional restrictions $\alpha = \beta = \epsilon = 0$ and $\mu = 2\gamma$: the unique solutions in this case are the Griffiths²⁰ metrics, containing as special cases the metrics^{3,4,19,41}. At first sight one would expect the class $\pi \neq 0$ to admit a larger set of solutions, but this is by no means guaranteed (compare with section 2.1): anyway no explicit examples appear to be known. The only property which is easy to demonstrate –though somewhat tedious to be included in the present paragraph– is that no solutions exist for which \mathbf{k} is non-expanding ($\Re(\rho) \neq 0$).

As a corollary of the above result we also obtain a new characterization of the class \mathcal{D} metrics¹⁰ with non-vanishing cosmological constant: if one of the multiple Weyl-PND's a Petrov type D Einstein-Maxwell solution would be non-aligned with a PND of \mathbf{F} then $R = 0$. In other words, for a non-vanishing cosmological constant, the double alignment property of the class \mathcal{D} metrics is a consequence of their Weyl-PND's being geodesic and shear-free.

It is tempting to conjecture that this same conclusion also will hold when $\Lambda = 0$. However it has not been possible so far to prove this, except for the special case of the Kundt space-times (i.e. in which the Weyl-PND \mathbf{k} has vanishing complex divergence). As the proof for this particular case is again quite tedious and little illuminating, I prefer to postpone this part to a possible later and more general publication.

V. DISCUSSION

Most results dealing with exact solutions for (algebraically special) Einstein-Maxwell fields have been obtained in the past introducing special coordinate systems, usually adapted to geodesic and/or shear-free null-congruences, or by imposing, sometimes haphazard looking, restrictions on the spin coefficients of a Newman-Penrose tetrad. This not only has turned their comparison and classification into an awkward procedure, often involving sophisticated computer algebra packages, but also has made it difficult for researchers and students entering the field to recognize the blanks which remain to be filled in. As furthermore the “Exact Solutions book”⁴⁰ touches the subject of non-aligned Einstein-Maxwell fields, or of aligned fields with a non-geodesic or shearing multiple PND, only superficially, I present below the known results, together with the ones obtained in the previous sections, schematically. From these diagrams I exclude the conformally flat case, as it implies³⁶ $\Lambda = 0$ with the only non-null member being therefore⁴⁰ the Bertotti-Robinson^{2,28,37} metric and the null members being given by a special class of plane waves^{1,30}.

In the figures below labels (1), (2), ... next to vertical arrows refer to results obtained in the corresponding previous sections, \nexists indicates that no solutions are allowed, while a question mark indicates that no solutions are known to occur in the literature. Roman capitals II, III, N, D refer to the Petrov types and ‘ $1 \times$ ’ or ‘ $2 \times$ ’ = indicate singly aligned or doubly aligned solutions.

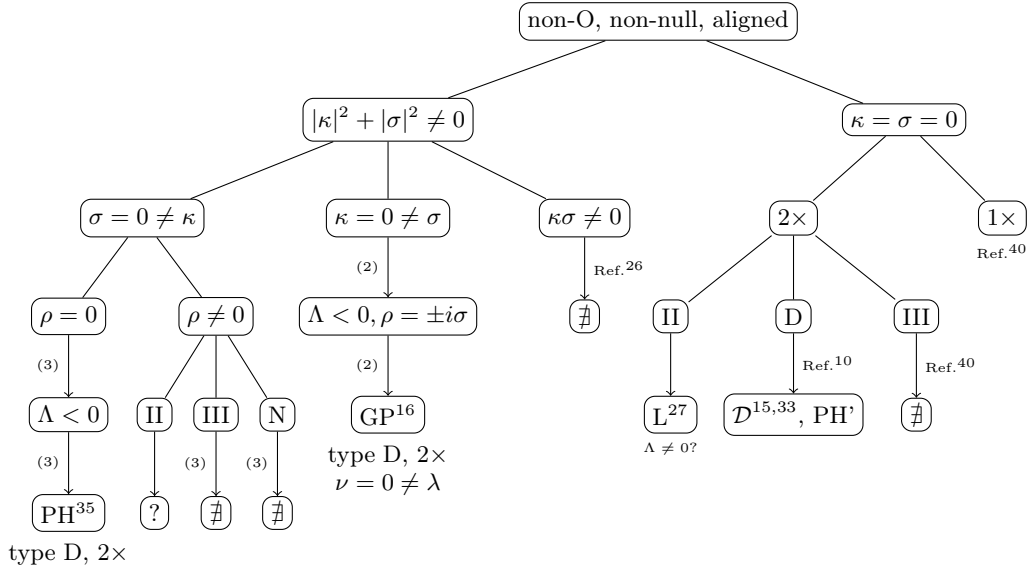


FIG. 1.

Algebraically special non-nul Einstein-Maxwell solutions for which the Weyl-PND \mathbf{k} is a PND of \mathbf{F} , the null-rotation about \mathbf{k} being chosen such that Φ_1 is the only non-vanishing component of the Maxwell spinor; L = Leroy, PH = Plebański-Hacyan, GP = Garcia-Plebański.

In Fig. 1 the sub-tree corresponding to the singly aligned Einstein-Maxwell solutions with a shear-free and geodesic multiple Weyl-PND (as well as the null fields, which automatically obey this condition) is not included, as little or no progress has been made in this area since the early nineties^{32,40}, with the bulk of the material contained in⁴⁰. The double occurrence of the Plebański-Hacyan metric in Fig. 1 (once as PH and once as PH', the GHP-primed version of PH with the roles of \mathbf{k} and ℓ interchanged) is due to the fact that PH is doubly aligned with only one null-ray being geodesic and both being shear-free. The absence of doubly aligned Petrov type III solutions in Fig. 1 has been noticed already in⁴⁰ for $\Lambda = 0$, but can easily be seen to hold also for $\Lambda \neq 0$.⁴⁰ also mentions that the Leroy metric²⁷ is the unique Einstein-Maxwell solution of Petrov type II with $\Lambda = 0$, in which both the Weyl PND's \mathbf{k} and ℓ are also PND's of \mathbf{F} and in which the multiple PND \mathbf{k} is geodesic and shear-free; this is not clear at all, as Leroy's solutions

were only obtained in the non-radiative sub-case. It furthermore remains to be checked whether this still holds –for a suitable generalisation of the Leroy metric– when $\Lambda \neq 0$.

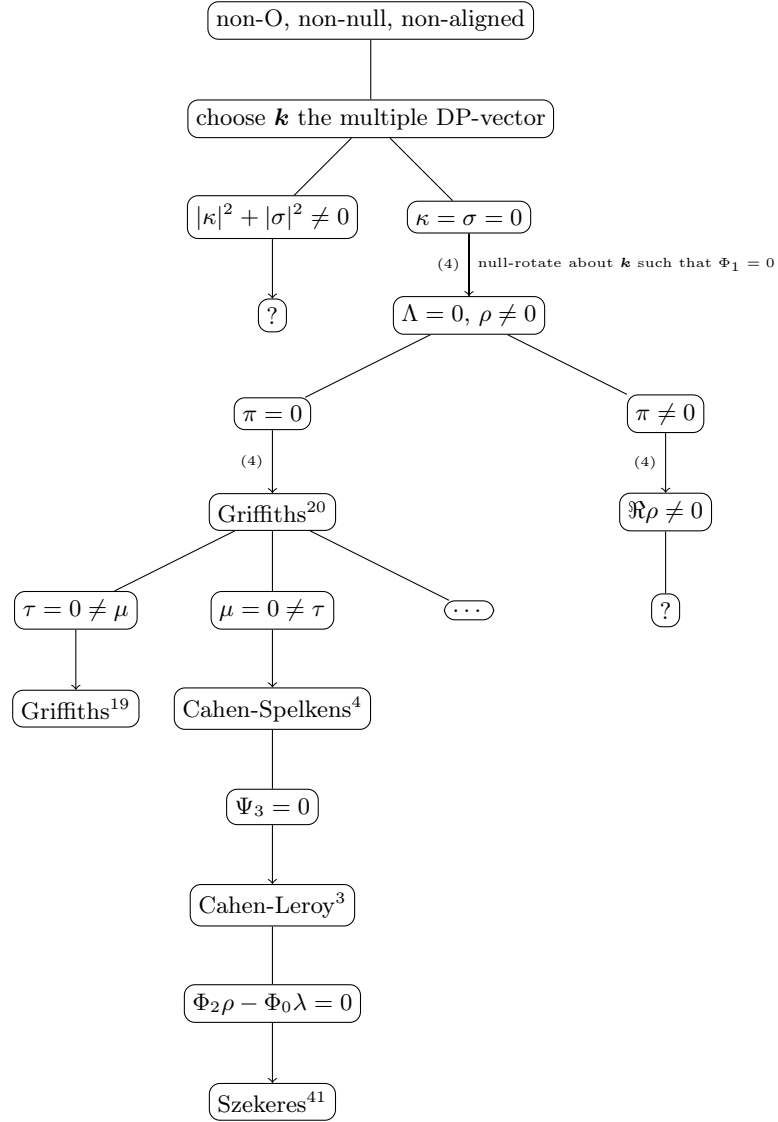


FIG. 2. Algebraically special non-nul Einstein-Maxwell solutions for which the multiple Weyl-PND is \mathbf{k} not a PND of \mathbf{F} . The sub-tree under Griffiths²⁰ presents in an invariant way the subcases mentioned in²⁰, originally obtained by imposing coordinate restrictions or restrictions on the NP spin-coefficients.

ACKNOWLEDGMENT

All calculations were done using the Maple symbolic algebra system⁴⁵. I also thank Lode Wylleman for a critical reading of the manuscript.

VI. APPENDIX: GHP, MAXWELL AND BIANCHI EQUATIONS

Weights of the spin-coefficients, the Maxwell and Weyl spinor components and the GHP operators:

$$\begin{aligned}\kappa &: [3, 1], \nu : [-3, -1], \sigma : [3, -1], \lambda : [-3, 1], \\ \rho &: [1, 1], \mu : [-1, -1], \tau : [1, -1], \pi : [-1, 1], \\ \Phi_0 &: [2, 0], \Phi_1 : [0, 0], \Phi_2 : [-2, 0], \\ \Psi_0 &: [4, 0], \Psi_1 : [2, 0], \Psi_2 : [0, 0], \Psi_3 : [-2, 0], \Psi_4 : [-4, 0], \\ \eth &: [1, -1], \eth' : [-1, 1], \mathbb{P}' : [-1, -1], \mathbb{P} : [1, 1].\end{aligned}$$

The prime operation is an involution with

$$\kappa' = -\nu, \sigma' = -\lambda, \rho' = -\mu, \tau' = -\pi, \quad (81)$$

$$\Psi_0' = \Psi_4, \Psi_1' = \Psi_3, \Psi_2' = \Psi_2, \quad (82)$$

$$\Phi_0' = -\Phi_2, \Phi_1' = -\Phi_1. \quad (83)$$

The GHP commutators acting on (p, q) weighted quantities are given by:

$$\begin{aligned}[\mathbb{P}, \mathbb{P}'] &= (\pi + \bar{\tau})\eth + (\bar{\pi} + \tau)\eth' + (\kappa\nu - \pi\tau + \frac{R}{24} - \Phi_{11} - \Psi_2)p \\ &\quad + (\bar{\kappa}\bar{\nu} - \bar{\pi}\bar{\tau} + \frac{R}{24} - \Phi_{11} - \bar{\Psi}_2)q, \quad (84)\end{aligned}$$

$$\begin{aligned}[\eth, \eth'] &= (\mu - \bar{\mu})\mathbb{P} + (\rho - \bar{\rho})\mathbb{P}' + (\lambda\sigma - \mu\rho - \frac{R}{24} - \Phi_{11} + \Psi_2)p \\ &\quad - (\bar{\lambda}\bar{\sigma} - \bar{\mu}\bar{\rho} - \frac{R}{24} - \Phi_{11} + \bar{\Psi}_2)q, \quad (85)\end{aligned}$$

$$[\mathbb{P}, \eth] = \bar{\pi}\mathbb{P} - \kappa\mathbb{P}' + \bar{\rho}\eth + \sigma\eth' + (\kappa\mu - \sigma\pi - \Psi_1)p + (\bar{\kappa}\bar{\lambda} - \bar{\pi}\bar{\rho} - \Phi_{01})q. \quad (86)$$

GHP equations:

$$\mathbb{P}\rho - \eth'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau + \kappa\pi + \Phi_{00}, \quad (87)$$

$$\mathbb{P}\sigma - \eth\kappa = (\rho + \bar{\rho})\sigma + (\bar{\pi} - \tau)\kappa + \Psi_0, \quad (88)$$

$$\mathbb{P}\tau - \mathbb{P}'\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + \Phi_{01} + \Psi_1, \quad (89)$$

$$\mathbb{P}\nu - \mathbb{P}'\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + \Psi_3 + \bar{\Phi}_1\Phi_2, \quad (90)$$

$$\eth\rho - \eth'\sigma = (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa + \Phi_{01} - \Psi_1, \quad (91)$$

$$\mathbb{P}'\sigma - \eth\tau = -\sigma\mu - \bar{\lambda}\rho - \tau^2 + \kappa\bar{\nu} - \Phi_{02}, \quad (92)$$

$$\mathbb{P}'\rho - \eth'\tau = -\bar{\mu}\rho - \lambda\sigma - \tau\bar{\tau} + \kappa\nu - \frac{R}{12} - \Psi_2. \quad (93)$$

Maxwell equations:

$$\mathbb{P}\Phi_1 - \eth'\Phi_0 = \pi\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2, \quad (94)$$

$$\mathbb{P}\Phi_2 - \eth'\Phi_1 = -\lambda\Phi_0 + 2\pi\Phi_1 + \rho\Phi_2. \quad (95)$$

Bianchi equations ($\Phi_{IJ} = \Phi_I\bar{\Phi}_J$ and $\Lambda = R/4 = \text{constant}$):

$$\begin{aligned}\eth'\Psi_0 - \mathbb{P}\Psi_1 + \mathbb{P}\Phi_{01} - \eth\Phi_{00} &= -\pi\Psi_0 - 4\rho\Psi_1 + 3\kappa\Psi_2 + \bar{\pi}\Phi_{00} + 2\bar{\rho}\Phi_{01} + 2\sigma\Phi_{10} \\ &\quad - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}, \quad (96)\end{aligned}$$

$$\begin{aligned}\mathbb{P}'\Psi_0 - \eth\Psi_1 + \mathbb{P}\Phi_{02} - \eth\Phi_{01} &= -\mu\Psi_0 - 4\tau\Psi_1 + 3\sigma\Psi_2 - \bar{\lambda}\Phi_{00} + 2\bar{\pi}\Phi_{01} + 2\sigma\Phi_{11} \\ &\quad + \bar{\rho}\Phi_{02} - 2\kappa\Phi_{12}, \quad (97)\end{aligned}$$

$$\begin{aligned}
3\bar{\partial}'\Psi_1 - 3\bar{\partial}\Psi_2 + 2\bar{\partial}\Phi_{11} - 2\bar{\partial}\Phi_{10} + \bar{\partial}'\Phi_{01} - \bar{\partial}'\Phi_{00} &= 3\lambda\Psi_0 - 9\rho\Psi_2 - 6\pi\Psi_1 + 6\kappa\Psi_3 + (\bar{\mu} - 2\mu)\Phi_{00} \\
&+ 2(\pi + \bar{\tau})\Phi_{01} + 2(\tau + \bar{\pi})\Phi_{10} + 2(2\bar{\rho} - \rho)\Phi_{11} + 2\sigma\Phi_{20} - \bar{\sigma}\Phi_{02} - 2\bar{\kappa}\Phi_{12} - 2\kappa\Phi_{21},
\end{aligned} \tag{98}$$

$$\begin{aligned}
3\bar{\partial}'\Psi_1 - 3\bar{\partial}\Psi_2 + 2\bar{\partial}\Phi_{12} - 2\bar{\partial}\Phi_{11} + \bar{\partial}'\Phi_{02} - \bar{\partial}'\Phi_{01} &= 3\nu\Psi_0 - 6\mu\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3 - \bar{\nu}\Phi_{00} \\
&+ 2(\bar{\mu} - \mu)\Phi_{01} - 2\bar{\lambda}\Phi_{10} + 2(\tau + 2\bar{\pi})\Phi_{11} + (2\pi + \bar{\tau})\Phi_{02} + 2(\bar{\rho} - \rho)\Phi_{12} + 2\sigma\Phi_{21} - 2\kappa\Phi_{22}.
\end{aligned} \tag{99}$$

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⁴³Some authors²¹ also use the name Plebański-Hacyan space-times for the exceptional class \mathcal{D} metrics obeying the condition $\frac{3}{2}\Psi_2 = |\Phi_1|^2$: these have $F = 0$ and hence have both \mathbf{k} and ℓ geodesic.

⁴⁴Already in²³ the authors mention that Kozarzewski's proof appeared to be erroneous, but this remark never seems to have attracted further attention.

⁴⁵This property is reminiscent of the result obtained in³⁶, according to which a conformally flat Einstein-Maxwell solution necessarily has a vanishing cosmological constant.

⁴⁶Code for manipulating the GHP equations can be downloaded from <http://users.ugent.be/~nvdbergh/rug/gr/ghp.mpl>